Differential Geometry I: Curves and surfaces

5.1 **(a)** Let $\gamma : I \to \mathbb{R}^n$ be a regular C^2 curve. Prove that it is biregular if and only if $\kappa_\gamma(t) \neq 0$.

(b) Let $\gamma : I \to \mathbb{R}^3$ be a Frenet regular curve. Prove that its torsion is *geometric*, i.e. for any C^3 reparametrization $h : J \to I$, if $\tilde{\gamma} = \gamma \circ h$ then we have

$$\tau_\gamma(h(u)) = \tau_{\tilde{\gamma}}(u) \quad \text{for all } \ u \in J.$$

5.2 Prove that the curve $\gamma(t) = (\cosh t, \sinh t, t)$ is biregular of class C^3, and compute its curvature vector, curvature (the curvature is the norm of the curvature vector) and torsion.

5.3 Consider the curve

$$\gamma(t) = (\cos t + t \sin t,\, \sin t - t \cos t,\, t^2), \quad t \in \mathbb{R}.$$

(a) Find the singular point(s) of this curve.

(b) Compute the arc length parameter $s = s(t)$ from the initial point $\gamma(0)$.

For the remaining questions, restrict to $t > 0$.

(c) Compute the tangent vector $T_\gamma(t)$ and the curvature vector $K_\gamma(t)$.

(d) Determine the biregular points of γ.

(e) Compute the curvature $\kappa_\gamma(t)$ and the principal normal vector $N_\gamma(t)$.

(f) Give the binormal vector $B_\gamma(t)$ (at biregular points).

(g) Compute the torsion of γ.

5.4 Let $\gamma : I \to \mathbb{R}^3$ be a Frenet regular curve. The *Darboux vector* of γ is the vector field along γ defined by

$$D_\gamma(u) := \tau_\gamma(u) T_\gamma(u) + \kappa_\gamma(u) B_\gamma(u).$$

Show that for any vector field A along γ written as $A(u) = a_1(u)T(u) + a_2(u)N(u) + a_3(u)B(u)$, we have

$$\frac{1}{V}\frac{dA}{du} = \frac{1}{V}(\dot{a}_1 T + \dot{a}_2 N + \dot{a}_3 B) + D \times A.$$

(This is the *Darboux formula*.)

5.5 Compute the Darboux vector of the right circular helix $\gamma(u) = (a \cos u,\, a \sin u,\, bu)$.

5.6 Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\gamma(t) = (t,\, t^2 + |t|^3,\, 0).$$

Show that this curve is *regular in the Frenet sense*, but not of class C^3. Then compute its Frenet frame.

5.7 What can be said about a (Frenet-regular) curve whose curvature and torsion are both constant?

5.8 Show that the torsion of a C^3 biregular curve $\gamma : I \to \mathbb{R}^3$ can be computed by the formula

$$\tau(u) = \frac{[\dot\gamma(u), \ddot\gamma(u), \dddot\gamma(u)]}{\|\dot\gamma(u) \times \ddot\gamma(u)\|^2} = \frac{[\dot\gamma, \ddot\gamma, \dddot\gamma]}{\kappa^2(u) V_\gamma^6(u)},$$

where $[x, y, z] = \langle x,\, y \times z\rangle$ denotes the mixed triple product of three vectors in $\mathbb{R}^3$.

5.9 Show that a C^3 biregular curve $\gamma : I \to \mathbb{R}^3$ is a right circular helix if and only if its Darboux vector is constant.

B. Additional Exercise

5.10 It is known that, up to rigid motion, the geometry of a curve is completely determined by its curvature and torsion. Hence, any geometric property can be expressed as one or more equations involving τ and κ. The goal of this exercise is to illustrate this fact in the case of *spherical curves* (i.e. curves lying on a sphere).

(a) Let $\gamma : I \to \mathbb{R}^3$ be a C^2 and regular curve parametrized by arc length and satisfying $\|\gamma(s)\| = r =$ constant. Prove that γ is biregular.

(b) Let $\gamma : I \to \mathbb{R}^3$ be a C^3 regular curve parametrized by arc length, satisfying $\|\gamma(s)\| = r =$ constant and such that it has nonzero torsion. Show that for all s,

$$\gamma(s) + \rho(s)N(s) + \frac{\dot\rho(s)}{\tau(s)}B(s) = 0,$$

where $\kappa(s)$ is the curvature of γ and $\rho(s) = \frac{1}{\kappa(s)}$ is the radius of curvature. Deduce that the function

$$s \mapsto \rho(s)^2 + \left(\frac{\dot\rho(s)}{\tau(s)}\right)^2$$

is constant.

(c) We will now establish (almost) the converse: Let $\gamma : I \to \mathbb{R}^3$ be a C^3 regular curve parametrized by arc length, with nonzero curvature and nonzero torsion. Assume, in addition the curvature satisfies $\dot{\kappa}_\gamma \neq 0$. Show that γ is a spherical curve (i.e. its trace lies on a sphere) if and only if

$$\rho(s)^2 + \left(\frac{\dot{\rho}(s)}{\tau(s)}\right)^2$$

is constant. Determine the center and radius of the sphere. (Note that the condition on the strict monotonicity of the curvature is necessary; find an example of a curve with $\kappa_\gamma = \text{const}$ which is not a spherical curve).